

Chapter 5

Modeling with Linear Programming

5.1 Introductory Example

SilComputers makes quarterly decisions about their product mix. While their full product line includes hundreds of products, we will consider a simpler problem with just two products: notebook computers and desktop computers. SilComputers would like to know how many of each product to produce in order to maximize profit for the quarter.

There are a number of limits on what SilComputers can produce. The major constraints are as follows:

1. Each computer (either notebook or desktop) requires a Processing Chip. Due to tightness in the market, our supplier has allocated 10,000 such chips to us.
2. Each computer requires memory. Memory comes in 16MB chip sets. A notebook computer has 16MB memory installed (so needs 1 chip set) while a desktop computer has 32MB (so requires 2 chip sets). We received a great deal on chip sets, so have a stock of 15,000 chip sets to use over the next quarter.
3. Each computer requires assembly time. Due to tight tolerances, a notebook computer takes more time to assemble: 4 minutes versus 3 minutes for a desktop. There are 25,000 minutes of assembly time available in the next quarter.

Given current market conditions, material cost, and our production system, each notebook computer produced generates \$750 profit, and each desktop produces \$1000 profit.

There are many questions SilComputer might ask. The most obvious are such things as “How many of each type computer should SilComputer produce in the next quarter?” “What is the maximum profit SilComputer can make?” Less obvious, but perhaps of more managerial interest are “How much should SilComputer be willing to pay for an extra memory chip set?” “What is the effect of losing 1,000 minutes of assembly time due to an unexpected machine failure?” “How much profit would we need to make on a 32MB notebook computer to justify its production?”

Linear programming gives us a mechanism for answering all of these questions quickly and easily. There are three steps in applying linear programming: modeling, solving, and interpreting.

5.1.1 Modeling

We begin by modeling this problem. Modeling a problem using linear programming involves writing it in the language of linear programming. There are rules about what you can and cannot do within

linear programming. These rules are in place to make certain that the remaining steps of the process (solving and interpreting) can be successful.

Key to a linear program are the *decision variables*, *objective*, and *constraints*.

Decision Variables. The decision variables represent (unknown) decisions to be made. This is in contrast to *problem data*, which are values that are either given or can be simply calculated from what is given. For this problem, the decision variables are the number of notebooks to produce and the number of desktops to produce. We will represent these unknown values by x_1 and x_2 respectively. To make the numbers more manageable, we will let x_1 be the number of 1000 notebooks produced (so $x_1 = 5$ means a decision to produce 5000 notebooks) and x_2 be the number of 1000 desktops. Note that a value like the quarterly profit is not (in this model) a decision variable: it is an outcome of decisions x_1 and x_2 .

Objective. Every linear program has an objective. This objective is to be either minimized or maximized. This objective has to be *linear* in the decision variables, which means it must be the sum of constants times decision variables. $3x_1 - 10x_2$ is a linear function. x_1x_2 is not a linear function. In this case, our objective is to maximize the function $750x_1 + 1000x_2$ (what units is this in?).

Constraints. Every linear program also has constraints limiting feasible decisions. Here we have four types of constraints: Processing Chips, Memory Sets, Assembly, and Nonnegativity.

In order to satisfy the limit on the number of chips available, it is necessary that $x_1 + x_2 \leq 10$. If this were not the case (say $x_1 = x_2 = 6$), the decisions would not be implementable (12,000 chips would be required, though we only have 10,000). Linear programming cannot handle arbitrary restrictions: once again, the restrictions have to be *linear*. This means that a linear function of the decision variables must be *related* to a constant, where *related* can mean less than or equal to, greater than or equal to, or equal to. So $3x_1 - 2x_2 \geq 10$ is a linear constraint, as is $-x_1 + x_3 = 6$. $x_1x_2 \leq 10$ is not a linear constraint, nor is $x_1 + 3x_2 < 3$. Our constraint for Processing Chips $x_1 + x_2 \leq 10$ is a linear constraint.

The constraint for memory chip sets is $x_1 + 2x_2 \leq 15$, a linear constraint.

Our constraint on assembly can be written $4x_1 + 3x_2 \leq 25$, again a linear constraint.

Finally, we do not want to consider decisions like $x_1 = -5$, where production is negative. We add the linear constraints $x_1 \geq 0$, $x_2 \geq 0$ to enforce nonnegativity of production.

Final Model. This gives us the complete model of this problem:

$$\begin{array}{ll} \text{Maximize} & 750x_1 + 1000x_2 \\ \text{Subject to} & \\ & x_1 + x_2 \leq 10 \\ & x_1 + 2x_2 \leq 15 \\ & 4x_1 + 3x_2 \leq 25 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

Formulating a problem as a linear program means going through the above process to clearly define the decision variables, objective, and constraints.

5.1.2 Solving the Model

Models are useful in their own right: they allow for a formal definition of a problem and can be useful in thinking about a problem. Linear programming models are particularly useful, however, because it is easy to have a computer solve for the optimal decision values.

For a model with only two variables, it is possible to solve the problem without a computer by drawing the feasible region and determining how the objective is optimized on that region. We go through that process here. The purpose of this exercise is to give you intuition and understanding of linear programming models and their solution. In any real application, you would use a computer to solve even two variable problems (we outline how to use Excel's Solver routine to find solutions in the next section).

We can represent a model with two variables by labeling the axes of a graph with each of the variables. The entire graph then represents possible decisions. Constraints are represented by lines on the graph, with the feasible region lying on one side of the line. The following figure illustrates this with the constraint $x_1 + x_2 \leq 10$.

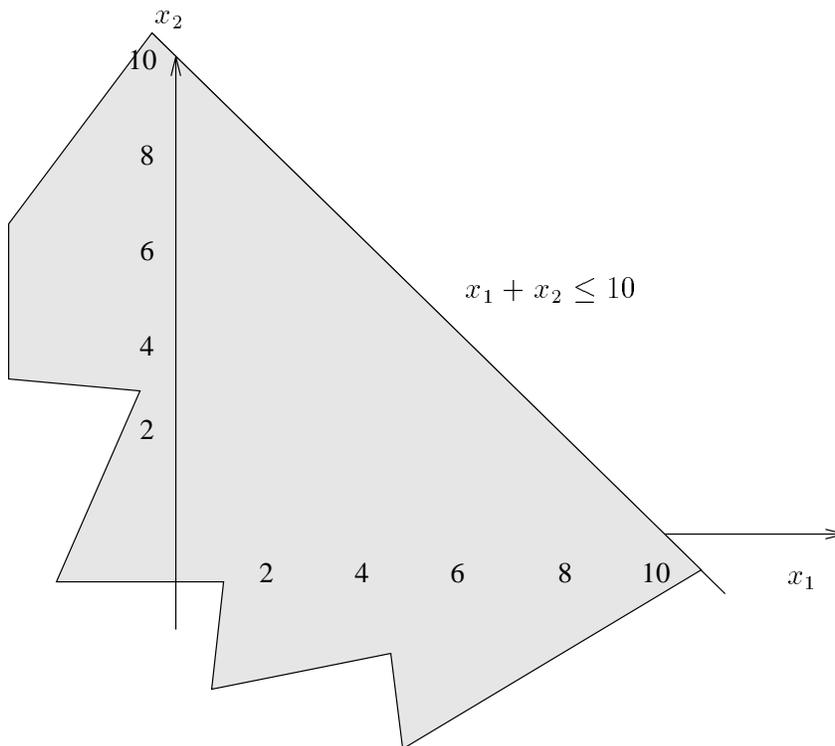


Figure 5.1: Single Constraint

We can continue this process and add in all of the constraints. Since every constraint must be satisfied, the resulting feasible region is the intersection of the feasible region for each constraint. This is shown in the following figure.

Note that just graphing the model gives us information we did not have before. It seems that the Chip constraint ($x_1 + x_2 \leq 10$) plays little role in this model. This constraint is dominated by other constraints.

Now, how can we find the optimal solution? We can include the objective function on this diagram by drawing *iso-profit* lines: lines along which the profit is the same. Since our goal is to maximize the profit, we can push the isoprofit line out until moving it any further would result in no feasible point (see diagram, z represents profit). Clearly the optimal profit occurs at point X .

What are the variable values at point X . Note that X is the intersection of the constraints:

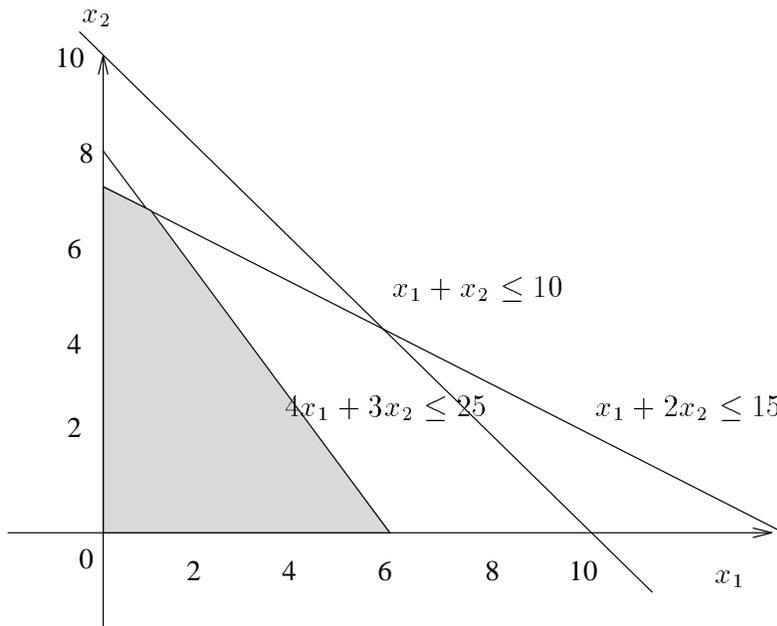


Figure 5.2: All Constraints

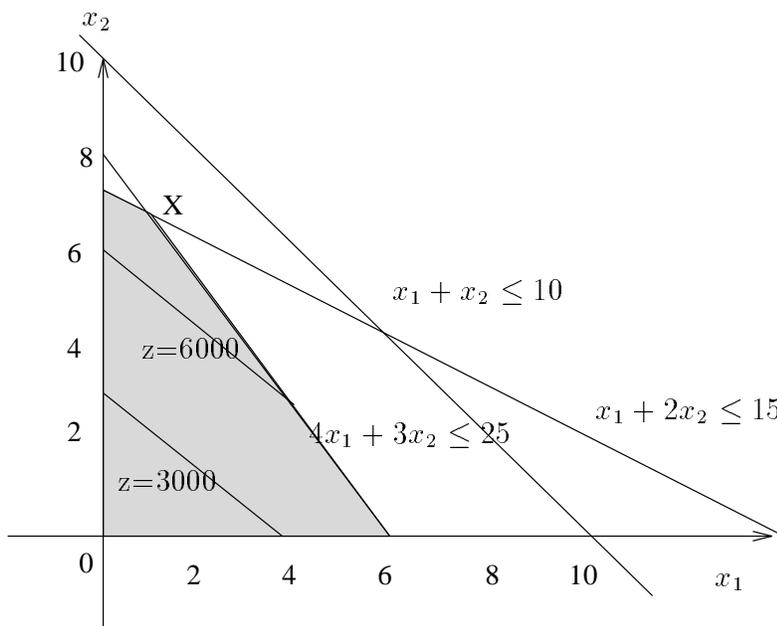


Figure 5.3: Finding Optimal Profit

$$\begin{aligned}x_1 + 2x_2 &= 15 \\4x_1 + 3x_2 &= 25\end{aligned}$$

We discussed in the previous section how to solve such equations: the solution here is $x_1 = 1$ and $x_2 = 7$. The optimal decision is to produce 1,000 notebooks and 7,000 desktops, for a profit of \$7,750,000.

5.1.3 Using Solver

Rather than the somewhat tedious and error-prone graphical method (which is limited to linear programs with 2 variables), special computer programs can be used to find solutions to linear programming models. The most widespread program is undoubtedly Solver, included in all recent versions of the Excel spreadsheet program. Solver, while not a state of the art code (which can cost upwards of \$15,000 per copy) is a reasonably robust, easy-to-use tool for linear programming. Solver uses standard spreadsheets together with an interface to define variables, objective, and constraints to define a linear program.

It is difficult to describe in words how to create a Solver spreadsheet, so we will do one or two in class. Here is a brief outline and some hints and shortcuts:

- We will start with a spreadsheet that has all of the data entered in some reasonably neat way.
- We will create the model in a separate part of the spreadsheet. We will have one cell for each variable. Solver will eventually put the optimal values in each cell.
- We will have a single cell to represent the objective. We will enter a formula that represents the objective. This formula must be a linear formula, so it must be of the form: $\text{cell1}*\text{cell1}' + \text{cell2}*\text{cell2}' + \dots$, where cell1 , cell2 and so on contain constant values and $\text{cell1}'$, $\text{cell2}'$ and so are the variable cells.

Helpful Hint: Excel has a function `sumproduct()` that is designed for linear programs. `sumproduct(a1..a10,b1..b10)` is identical to $a1*b1+a2*b2+a3*b3+\dots+a10*b10$. This function will save much time and aggravation. All that is needed is that the length of the first range is the same as the length of the second range (so one can be horizontal and the other vertical).

Helpful Hint: It is possible to assign names to cells and ranges (under the Insert-Name menu). Rather than use `a1..a10` as the variables, you can name that range `var` (for example) and then use `var` wherever `a1..a10` would have been used.

- We then have a cell to represent the left hand side of each constraint (again a linear function) and another cell to represent the right hand side (a constant).
- We then select Solver under the Tools menu. This gives a form to fill out to define the linear program.
- In the “Set Cell” box, select the objective cell. Choose Maximize or Minimize.
- In the “By Changing Cells”, put in the range containing the variable cells.

- We next add the constraints. Press the “Add...” button to add constraints. The dialog box has three parts for the left hand side, the type of constraint, and the right hand side. Put the cell references for a constraint in the form, choose the right type, and press “Add”. Continue until all constraints are added. On the final constraint, press “OK”.
- We need to explicitly include nonnegativity constraints.

Helpful Hint: It is possible to include ranges of constraints, as long as they all have the same type. $c1..e1 \leq c3..e3$ means $c1 \leq c3, d1 \leq d3, e1 \leq e3$. $a1..a10 \geq 0$ means each individual cell must be greater than or equal to 0.

- Push the options button and toggle the “Assume Linear Model” in the resulting dialog box. This tells Excel to call a linear rather than a nonlinear programming routine so as to solve the problem more efficiently. This also gives you sensitivity ranges, which are not available for nonlinear models.
- Push the Solve button. In the resulting dialog box, select “Answer” and “Sensitivity”. This will put the answer and sensitivity analysis in two new sheets. Ask Excel to “Keep Solver values”, and your worksheet will be updated so that the optimal values are in the variable cells.

5.2 Second Example: Market Allocations

MegaMarketing is planning a concentrated one week advertising campaign for their new CutsEverything SuperKnife. The ads have been designed and produced and now they wish to determine how much money to spend in each advertising outlet. In reality, they have hundreds of possible outlets to choose from. We will illustrate their problem with two outlets: Prime-time TV, and newsmagazines.

The problem of optimally spending advertising dollars can be formulated in many ways. For instance, given a fixed budget, the goal might be to maximize the number of target customers reached (a target customer is a customer with a reasonable chance of purchasing the product).

An alternative approach, which we adopt here, is to define targets for reaching each market segment and to minimize the money spent to reach those targets. For this product, the target segments are Teenage Boys, Affluent Women (ages 40-49), and Retired Men. Each minute of primetime TV and page of newsmagazine advertisement reaches the following number of people (in millions):

Outlet	Boys	Women	Men	Cost
TV	5	1	3	600
Mag	2	6	3	500
Target	24	18	24	

Again, MegaMarketing is interested in straightforward answers like how many units of each outlet to purchase to meet the segment goals. They are also interested in such questions as “How much will it cost to reach an extra million retired men?”, “One radio spot reaches 1 million boys, 1 million women, and 1 million men: how much are we willing to pay for such a spot?”, and similar questions.

5.2.1 Modeling

The process of modeling using linear programming is fairly consistent: define the decision variables, objective, and constraints.

Decision Variables. In this case, the decision variables are the number of units of each outlet to purchase. Let x_1 be the number of TV spots, and x_2 the number of magazine pages.

Objective. Our objective is to minimize the amount we spend: $600x_1 + 500x_2$.

Constraints. We have one constraint for each market segment: we must be certain that we reach sufficient people of each segment. For boys, this corresponds to the constraint:

$$5x_1 + 2x_2 \geq 24$$

Similar constraints for the other segments gives us the full formulation as:

$$\begin{array}{ll} \text{Minimize} & 600x_1 + 500x_2 \\ \text{Subject to} & \\ & 5x_1 + 2x_2 \geq 24 \\ & x_1 + 6x_2 \geq 18 \\ & 3x_1 + 3x_2 \geq 24 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

5.2.2 Solving the Model

Since this model has only two variables, we can solve the problem in two ways: graphically and using SOLVER (with SOLVER being much easier). We start by solving this graphically.

The first step is to graph the feasible region, as given in the following figure:

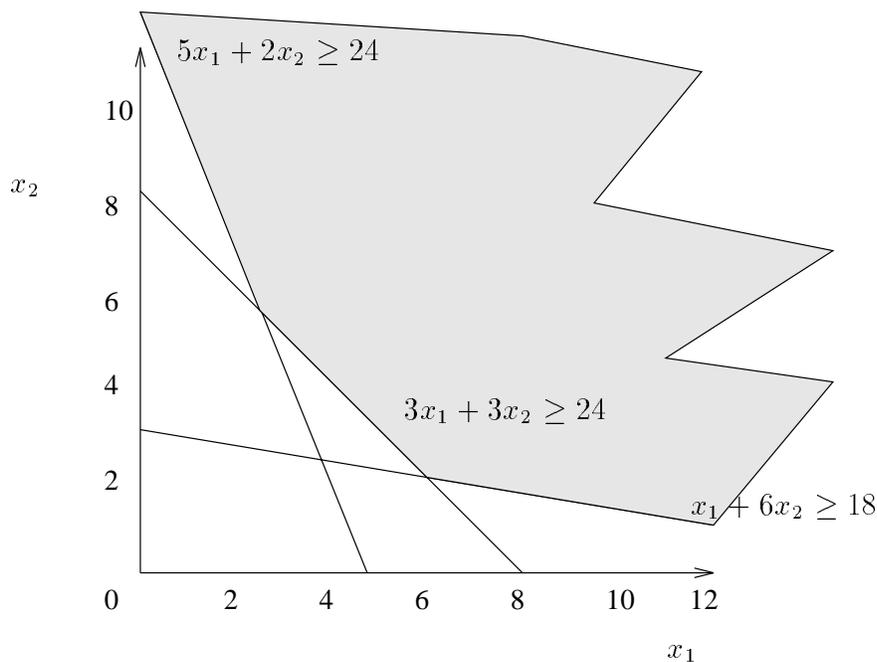


Figure 5.4: Marketing example

The next step is to put some iso-cost lines on the diagram: lines representing points that have the same cost. We mark the optimal point with an X .

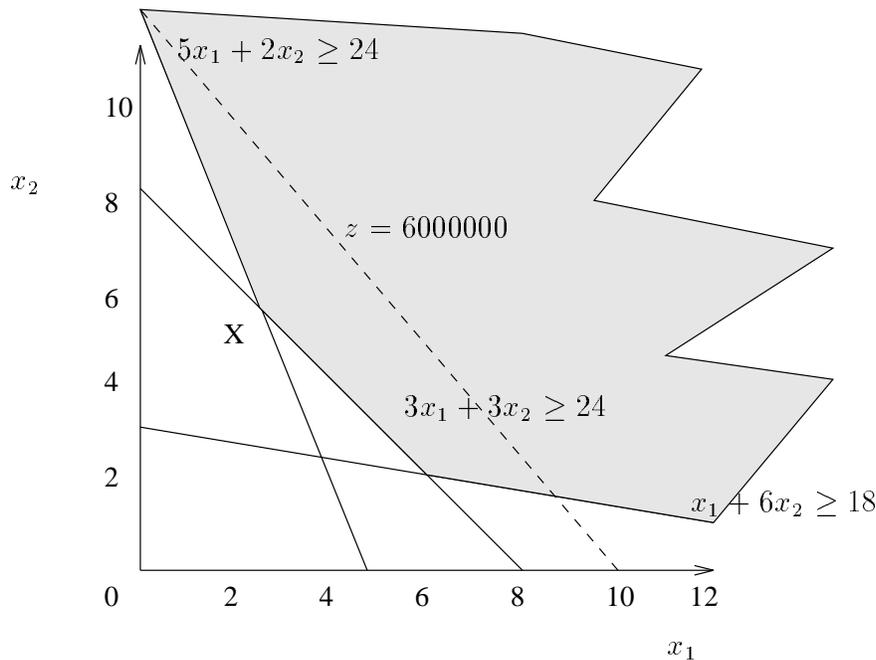


Figure 5.5: Marketing example with Isocost Line

X is the intersection of the constraints:

$$\begin{aligned} 5x_1 + 2x_2 &= 24 \\ 3x_1 + 3x_2 &= 24 \end{aligned}$$

The optimal solution is $x_1 = 2.67$ and $x_2 = 5.33$

5.2.3 Using Solver

The SOLVER model for this problem is on the course web pages.

5.3 Common Features

Hidden in our models of these problems are a number of assumptions. The usefulness of a model is directly related to how close reality matches up with these assumptions.

The first assumption is due to the linear form of our functions. Since the objective is linear, the contribution to the objective of any decision variable is proportional to the value of the decision variable. Producing twice as much of a product produces twice as much profit; buying twice as many pages of ads costs twice as much. This is the *Proportionality Assumption*.

Furthermore, the contribution of a variable to the objective is independent of the values of the other variables. One notebook computer is worth \$750, independent of how many desktop computers we produce. This is the *Additivity Assumption*.

Similarly, since each constraint is linear, the contribution of each variable to the left hand side of each constraint is proportional to the value of the variable and independent of the values of any other variable.

These assumptions are quite restrictive. We will see, however, that clever modeling can handle situations that may appear to violate these assumptions.

The next assumption is the *Divisibility Assumption*: it is possible to take any fraction of any variable. Rethinking the Marketing example, what does it mean to purchase 2.67 television ads? It may be that the divisibility assumption is violated in this example. Or, it may be that the units are such that 2.67 “ads” actually corresponds to 2666.7 minutes of ads, in which case we can “round off” our solution to 2667 minutes with little doubt that we are getting an optimal or nearly optimal solution. Similarly, a fractional production quantity may be worisome if we are producing a small number of battleships or be innocuous if we are producing millions of paperclips. If the Divisibility Assumption is important and does not hold, then a technique called *integer programming* rather than linear programming is required. This technique takes orders of magnitude more time to find solutions but may be necessary to create realistic solutions. You will learn more about this in 45-761.

The final assumption is the *Certainty Assumption*: linear programming allows for no uncertainty about the numbers. An ad will reach the given number of people; the number of assembly hours we give will certainly be available.

It is very rare that a problem will meet all of the assumptions exactly. That does not negate the usefulness of a model. A model can still give useful managerial insight even if reality differs slightly from the rigorous requirements of the model. For instance, the knowledge that our chip inventory is more than sufficient holds in our first model even if the proportionality assumptions are not satisfied completely.

5.4 Linear Programming Models

Linear programming models are found in almost every field of business (and beyond!). The next sections go through a number of problems, showing how to model them by the appropriate choice of decision variables, objective, and constraints. In all cases, we will describe the problem and give a model. Solver worksheets for each of these is available on the web page.

5.4.1 Diet Problem

Problem Definition

What is the perfect diet? An ideal diet would meet or exceed basic nutritional requirements, be inexpensive, have variety, and be “pleasing to the palate”. How can we find such a diet?

Suppose the only foods in the world are as follows:

Food	Serving Size	Energy (kcal)	Protein (g)	Calcuim (mg)	Price (cents/serving)	Limit (servings/day)
Oatmeal	28g	110	4	2	3	4
Chicken	100g	205	32	12	24	3
Eggs	2 Large	160	13	54	13	2
Whole Milk	237cc	160	8	285	9	8
Cherry Pie	170g	420	4	22	20	2
Pork&Beans	260g	260	14	80	19	2

After consulting with nutritionists, we decree that a satisfactory diet has at least 2000 kcal of energy, 55 g of protein, and 800 mg of calcium (vitamins and iron are supplied by pills). While some of us would be happy to subsist on 10 servings of pork and beans, we have decided to impose variety by having a limit on the number of servings/day for each of our six foods. What is the least cost satisfactory diet?

Problem Modeling

First we decide on decision variables. Let us label the foods 1, 2, \dots , 6, and let x_i represent the number of servings of food i in the diet.

Our objective is to minimize cost, which can be written

$$3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6$$

We have constraints for energy, protein, calcium, and for each serving/day limit. This gives the formulation:

$$\begin{array}{ll} \text{Minimize} & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\ \text{Subject to} & \\ & 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \geq 2000 \\ & 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \geq 55 \\ & 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \geq 800 \\ & x_1 \leq 4 \\ & x_2 \leq 3 \\ & x_3 \leq 2 \\ & x_4 \leq 8 \\ & x_5 \leq 2 \\ & x_6 \leq 2 \\ & x_i \geq 0 \text{ (for all } i) \end{array}$$

Discussion

The creation of optimal diets was one of the first uses of linear programming. Some difficulties with linear programming include difficulties in formulating “palatability” requirements, and issues of divisibility (no one wants to eat half a green bean). These linear programming models do give an idea on how much these palatability requirements are costing.

5.4.2 Workforce Planning

Problem Definition.

Consider a restaurant that is open seven days a week. Based on past experience, the number of workers needed on a particular day is given as follows:

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Number	14	13	15	16	19	18	11

Every worker works five consecutive days, and then takes two days off, repeating this pattern indefinitely. How can we minimize the number of workers that staff the restaurant?

Model.

A natural (and wrong!) first attempt at this problem is to let x_i be the number of people working on day i . Note that such a variable definition does not match up with what we need to find. It does us no good to know that 15 people work Monday, 13 people Tuesday, and so on because it does not tell us how many workers are needed. Some workers will work both Monday and Tuesday, some only one day, some neither of those days. Instead, let the days be numbers 1 through 7 and let x_i be the number of workers who begin their five day shift on day i . Our objective is clearly:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

Consider the constraint for Monday's staffing level of 14. Who works on Mondays? Clearly those who start their shift on Monday (x_1). Those who start on Tuesday (x_2) do not work on Monday, nor do those who start on Wednesday (x_3). Those who start on Thursday (x_4) do work on Monday, as do those who start on Friday, Saturday, and Sunday. This gives the constraint:

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 14$$

Similar arguments give a total formulation of:

$$\begin{array}{ll} \text{Minimize} & \sum_i x_i \\ \text{Subject to} & \\ & x_1 + x_4 + x_5 + x_6 + x_7 \geq 14 \\ & x_1 + x_2 + x_5 + x_6 + x_7 \geq 13 \\ & x_1 + x_2 + x_3 + x_6 + x_7 \geq 15 \\ & x_1 + x_2 + x_3 + x_4 + x_7 \geq 16 \\ & x_1 + x_2 + x_3 + x_4 + x_5 \geq 19 \\ & x_2 + x_3 + x_4 + x_5 + x_6 \geq 18 \\ & x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 \\ & x_i \geq 0 \text{ (for all } i) \end{array}$$

Discussion.

Workforce modeling is a well developed area. Note that our model has only one type of shift, but the model is easily extended to other types of shifts, with differing shift costs.

5.4.3 Financial Portfolio**Problem Definition.**

In your finance courses, you will learn a number of techniques for creating "optimal" portfolios. The optimality of a portfolio depends heavily on the model used for defining risk and other aspects of financial instruments. Here is a particularly simple model that is amenable to linear programming techniques.

Consider a mortgage team with \$100,000,000 to finance various investments. There are five categories of loans, each with an associated return and risk (1-10, 1 best):

Loan/investment	Return (%)	Risk
First Mortgages	9	3
Second Mortgages	12	6
Personal Loans	15	8
Commercial Loans	8	2
Government Securities	6	1

Any uninvested money goes into a savings account with no risk and 3% return. The goal for the mortgage team is to allocate the money to the categories so as to:

- Maximize the average return per dollar
- Have an average risk of no more than 5 (all averages and fractions taken over the invested money (not over the saving account)).
- Invest at least 20% in commercial loans
- The amount in second mortgages and personal loans combined should be no higher than the amount in first mortgages.

Model

Let the investments be numbered 1...5, and let x_i be the amount invested in investment i . Let x_s be the amount in the savings account. The objective is to maximize

$$9x_1 + 12x_2 + 15x_3 + 8x_4 + 6x_5 + 3x_s$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_s = 100,000,000.$$

Now, let's look at the average risk. Since we want to take the average over only the invested amount, a direct translation of this constraint is

$$\frac{3x_1 + 6x_2 + 8x_3 + 2x_4 + x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \leq 5$$

This constraint is not linear, but we can cross multiply, and simplify to get the equivalent linear constraint:

$$-2x_1 + x_2 + 3x_3 - 3x_4 - 4x_5 \leq 0$$

Similarly we need

$$x_4 \geq 0.2(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$-0.2x_1 - 0.2x_2 - 0.2x_3 + 0.8x_4 - 0.2x_5 \geq 0$$

The final constraint is

$$x_2 + x_3 - x_1 \leq 0$$

Together with nonnegativity, this gives the entire formulation.

Discussion

Optimal portfolios do not just happen: they must be calculated, and there is a constant interplay between models and solvability. Linear programming models provide great modeling power with a great limit: the handling of risk must be done in a linear fashion (like our Risk factors here). Other models you will see in finance will look at the co-variance of returns between investments, a fundamentally nonlinear effect. This can give rise to nonlinear models like those that try to minimize variance subject to return requirements. It is very difficult to embed idiosyncratic constraints (like (c) and (d) here) in such models.

5.4.4 Production over Time

Problem Description

Many problems are multiperiod, where a series of decisions have to be made over time. For instance, in our very first SilComputer example, production decisions for computers have to be made with the future in mind. In this example, it would be easy to work with both desktop and notebook computers but let's work just with notebook computers.

SilComputer needs to meet the demand of its largest corporate and educational customers for notebook computers over the next four quarters (before its current model becomes obsolete). SilComputer currently has 5,000 notebook computers in inventory. Expected demand over the next four quarters for its notebook is 7,000; 15,000; 10,000; and 8,000. SilComputer has sufficient capacity and material to produce up to 10,000 computers in each quarter at a cost of \$2000 per notebook. By using overtime, up to an additional 2,500 computers can be produced at a cost of \$2200 each. Computers produced in a quarter can be used either to meet that quarter's demand, or be held in inventory for use later. Each computer in inventory is charged \$100 to reflect carrying costs.

How should SilComputer meet its demand for notebooks at minimum cost?

Modeling.

The decisions seem to be how many of each computer to produce in each period at regular time, how many to produce at overtime, and how much inventory to carry in each period. Let's denote our time periods $t = 1, 2, 3, 4$. Let x_t be the number of notebooks produced in period t at regular time and y_t be the number of notebooks produced in period t at overtime. Finally, let i_t be the inventory at the end of period t .

Look at the first quarter: how are these variables restricted and related?

Clearly we need $x_1 \leq 10000$ and $y_1 \leq 2500$. Now, anything that starts as inventory or is produced in the period must either be used to meet demand or ends up as inventory at the end of period 1. This means:

$$5000 + x_1 + y_1 = 7000 + i_1$$

For period 2, in addition to the upper bounds, we get the constraint

$$i_1 + x_2 + y_2 = 15000 + i_2$$

For period 3, we get

$$i_2 + x_3 + y_3 = 10000 + i_3$$

and for period 4 (assuming no inventory at the end):

$$i_3 + x_4 + y_4 = 8000$$

Our objective charges 2000 for each x , 2200 for each y , and 100 for each i :

$$2000x_1 + 2000x_2 + 2000x_3 + 2000x_4 + 2200y_1 + 2200y_2 + 2200y_3 + 2200y_4 + 100i_1 + 100i_2 + 100i_3$$

Discussion.

The constraints tracking where production goes are known as *conservation* constraints, and model the requirement that production never disappears. Whenever you have multiple types of variables in a formulation, be certain to think about whether there are necessary constraints that link them.

5.4.5 Investing over Time

We are going to manage an investment portfolio over a 6-year time horizon. We begin with \$1000, and at various times we can invest in one or more of the following:

Savings account X, annual yield 5%.

Security Y, 2-year maturity, total yield 12% if bought now, 11% thereafter.

Security Z, 3-year maturity, total yield 18%.

Security W, 4-year maturity, total yield 24%.

To keep things simple we will assume that each security can be bought in any denomination. (This assumption can be relaxed if one uses *integer* or *dynamic* programming .) We can make savings deposits or withdrawals anytime. We can buy Security Y any year but year 3. We can buy Security Z anytime after the first year. Security W, now available, is a one-time opportunity.

We let x_t be the amount of money invested in the savings account X at the beginning of year t , and similarly for y_t , z_t , and w_t . We will put any money not tied up in securities into the savings account. The situation is summed up in the figure.

The problem is really a kind of inventory problem. In a given year, the amount of money carried forward from the previous year (in savings), plus the yield from securities that mature that year, equals the amount invested in new securities plus the amount of money left over for next year. There is one twist: inventory grows while in storage. (In real inventory problems, stock often decreases over time due to spoilage, etc., and this can be reflected in negative “interest” rates.) Let v be the final yield when all securities are cashed in at the end of the sixth year. Then if the objective is to maximize final yield, the LP is,

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & x_1 + y_1 + w_1 = 1000 \\ & x_2 + y_2 + z_2 = 1.05x_1 \\ & x_3 + z_3 = 1.05x_2 + 1.12y_1 \\ & x_4 + y_4 + z_4 = 1.05x_3 + 1.11y_2 \\ & x_5 + y_5 = 1.05x_4 + 1.18z_2 + 1.24w_1 \\ & x_6 = 1.05x_5 + 1.11y_4 + 1.18z_3 \\ & v = 1.05x_6 + 1.11y_5 + 1.18z_4, \\ & x_t, y_t, z_t, w_t \geq 0 \end{aligned}$$

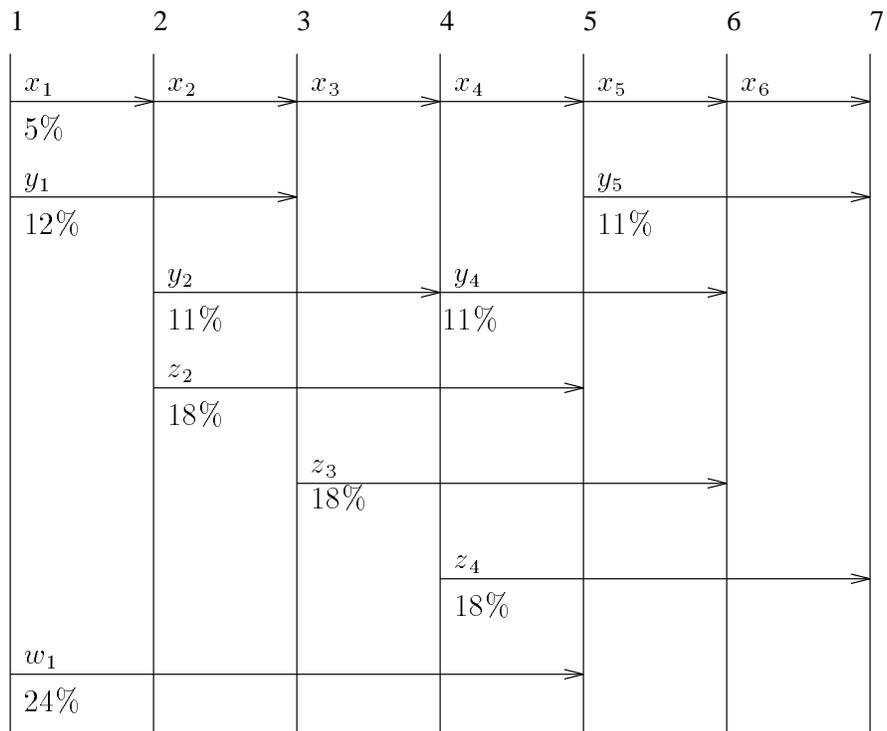


Figure 5.6: Possible investments over a 6-year horizon

The final assets come to \$1387.6, an average annual yield of 5.61%

5.4.6 Problems

Modeling is an art, and it takes practice. The following examples show the variety of problems that can be attacked by linear programming, and give you the opportunity to try your hand at some problems.

Exercise 44 A plant produces two types of refrigerators, A and B. There are two production lines, one dedicated to producing refrigerators of Type A, the other to producing refrigerators of Type B. The capacity of the production line for A is 60 units per day, the capacity of the production line for B is 50 units per day. A requires 20 minutes of labor whereas B requires 40 minutes of labor. Presently, there is a maximum of 40 hours of labor per day which can be assigned to either production line. Profit contributions are \$20 per refrigerator of Type A produced and \$30 per Type B produced. What should the daily production be? Solve graphically and by Solver.

Exercise 45 Albert, Bill, Charles, David, and Edward have gotten into a bind. After a series of financial transactions, they have ended up each owing some of the others huge amounts of money. In fact, near as the lawyers can make out, the debts are as follows

Debtor	Creditor	Amount (\$millions)
A	E	10
A	C	3
B	A	5
C	B	6
C	D	4
D	A	4
E	C	7
E	D	3

The question is, who is bankrupt? We will say that a person i is bankrupt if there is no possible transfer of funds among the people such that i completely pays off his obligations, and the transfer of funds satisfies the following condition: for every two persons j and k , the amount paid by person j to person k is no greater than the debt of j to k . For instance, Albert is bankrupt since he owes 10, and is only owed 9. Formulate the problem of determining whether Bill is bankrupt as a linear program. Then modify your formulation to determine if each of the others is bankrupt. *This example may look contrived, but it is inspired by a solution to the debts involved in a crash of Kuwait's al-Mankh stock market.*

Exercise 46 Due to an unexpected glut of orders, Blaster Steel has decided to hire temporary workers for a five day period to clear out the orders. Each temporary worker can work either a two day shift or a three day shift for this period (shifts must be consecutive days). At least 10 workers are needed on days 1, 3, 5, and at least 15 workers are needed on days 2 and 4. A worker on a two day shift gets paid \$125/day, while those on a three day shift gets paid \$100/day.

(a) Formulate the problem of hiring temporary workers to minimize cost while meeting the demand for workers.

(b) Due to a limited number of training personnel, no more than 10 workers can start their shift on any day. Update your formulation in (a) to take this into account.

(c) Union regulations require that at least half of all money spent on workers go to those who work three day shifts. Update your formulation in (a) to handle this requirement.

(d) There are four people who are willing to work a shift consisting of days 1, 2, and 5, for a payment of \$110/day. Update your formulation in (a) to handle this possibility.

Exercise 47 You have \$1000 to invest in Securities 1 and 2; uninvested funds are deposited in a savings account with annual yield of 5%. You will sell the securities when they reach maturity. Security 1 matures after 2 years with a total yield of 12%. Security 2 matures after 3 years with a total yield of 19%. Your planning horizon is 7 years, and you want to maximize total assets at the end of the seventh year. Suppose either security may be purchased in arbitrarily small denominations.

- a) Let x_{it} be the amount invested in Security i at the beginning of year t ; the savings account can be considered Security 0. Write the appropriate LP model.
- b) Solve the problem by computer and indicate your optimal portfolio in each year 1,...,7.

Exercise 48 Chemco produces two chemicals: A and B . These chemicals are produced via two manufacturing processes. Process 1 requires 2 hours of labor and 1 lb. of raw material to produce 2 oz. of A and 1 oz. of B . Process 2 requires 3 hours of labor and 2 lb. of raw material to produce 3 oz. of A and 2 oz. of B . Sixty hours of labor and 40 lb. of raw material are available. Chemical A sells for \$16 per oz. and B sells for \$14 per oz. Formulate a linear program that maximizes Chemco's revenue.

Hint: Define the amounts of Process 1 and Process 2 used, as your decision variables.

Exercise 49 Gotham City National Bank is open Monday-Friday from 9 am to 5 pm. From past experience, the bank knows that it needs the following number of tellers.

Time Period	Tellers Required
9 – 10	4
10 – 11	3
11 – noon	4
noon – 1	6
1 – 2	5
2 – 3	6
3 – 4	8
4 – 5	8

The bank hires two types of tellers. Full-time tellers work 9-5 five days a week, except for 1 hour off for lunch, either between noon and 1 pm or between 1 pm and 2 pm. Full-time tellers are paid (including fringe benefits) \$25/hour (this includes payment for lunch hour). The bank may also hire up to 3 part-time tellers. Each part-time teller must work exactly 4 consecutive hours each day. A part-time teller is paid \$20/hour and receives no fringe benefits. Formulate a linear program to meet the teller requirements at minimum cost.

Exercise 50 Sales forecasts for the next four months are (in thousand of units):

October	10
November	16
December	10
January	12

September's production was set at 12,000 units. Varying production rate incurs some cost: production can be increased from one month to the next at a cost of \$2 per unit and it can be

decreased at a cost of \$0.50 per unit. In addition, inventory left at the end of a month can be stored at a cost of \$1 per unit per month. Given current demand, there will be no inventory at the end of September. No inventory is desired at the end of January. Formulate a linear program that minimizes the total cost (varying production rate + inventory costs) of meeting the above demand.

Exercise 51 An oil company blends gasoline from three ingredients: butane, heavy naphtha and catalytic reformat. The characteristics of the ingredients as well as minimum requirements for regular gasoline are given below:

	Catalytic		Heavy	Gasoline
	Butane	Reformat	Naphtha	
Octane	120	100	74	≥ 89
Vapor Pressure	60	2.5	4.0	≤ 11
Volatility	105	3	12	≥ 17

The cost (per gallon) of butane is \$0.58, it is \$1.55 for catalytic reformat and \$0.85 for heavy naphtha. How many gallons of the three ingredients should be blended in order to produce 12,000 gallons of gasoline at minimum cost?

Exercise 52 An electric utility has six power plants on the drawing board. The anticipated useful life of these plants is 30 years for the coal-fired plants (plants 1, 2 and 3) and 40 years for the fuel-fired plants (plants 4, 5 and 6). Plants 1, 2 and 4 will be on line in year 5. Plants 3 and 5 in year 15. Plant 6 in year 25. The cost of installing generating capacity at Plant i , discounted to the present, is c_i per megawatt, for $i = 1, 2, \dots, 6$. Projected power demand in year t is D_t megawatts, for $t = 10, 20, 30, 40$. Due to environmental regulations, the fraction of generating capacity at coal-fired plants in year t , relative to total generating capacity in year t , can be at most r_t for $t = 10, 20, 30, 40$.

The generating capacity of each plant has yet to be determined. Write a linear program that assigns capacities to the six plants so as to minimize total present cost of installing generating capacity, while meeting demand and satisfying the environmental constraints in years $t = 10, 20, 30, 40$.

Exercise 53 (Wagner) You must decide how many tons x_1 of pure steel and how many tons x_2 of scrap metal to use in manufacturing an alloy casting. Pure steel costs \$300 per ton and scrap \$600 per ton (larger because the impurities must be skimmed off). The customer wants at least 5 tons but will accept a larger order, and material loss in melting and casting is negligible. You have 4 tons of steel and 7 tons of scrap to work with, and the weight ratio of scrap to pure cannot exceed $7/8$ in the alloy. You are allotted 18 hours melting and casting time in the mill; pure steel requires 3 hrs per ton and scrap 2 hrs per ton.

- Write a linear programming model for the problem assuming that the objective is to minimize the total steel and scrap costs.
- Graph the problem obtained in (a).
- Solve the problem in (a) using Solver and indicate the optimal values of x_1, x_2 . What is the weight ratio of scrap to pure steel?

Exercise 54 (Wagner) An airline must decide how many new flight attendants to hire and train over the next six months. The staff requirements in person-flight-hours are respectively 8000, 9000, 7000, 10,000, 9000, and 11,000 in the months January through June. A new flight attendant is

trained for one month before being assigned to a regular flight and therefore must be hired a month before he or she is needed. Each trainee requires 100 hours of supervision by experienced flight attendants during the training month, so that 100 fewer hours are available for flight service by regular flight attendants.

Each experienced flight attendant can work up to 150 hours a month, and the airline has 60 regular flight attendants available at the beginning of January. If the maximum time available from experienced flight attendants exceeds a month's flying and training requirements, they work fewer hours, and none are laid off. At the end of each month, 10% of the experienced flight attendants quit their jobs.

An experienced flight attendant costs the company \$1700 and a trainee \$900 a month in salary and benefits.

- a) Formulate the problem as a linear programming model. Let x_t be the number of flight attendants that begin training in month t . Hint. Let y_t be the number of experienced flight attendants available in month t ; also let $x_0 = 0$, $y_1 = 60$. The problem is really an inventory problem, with two kinds of stock: trainees and experienced employees. The "holding costs" are the salaries, and the demands are the number of flight hours needed. "Spoilage" is attrition of experienced staff.
- b) Solve the model with Solver and indicate the solution value for the x_t 's and y_t 's.

Exercise 55 (Wagner) A lumber company operates a sawmill that converts timber to lumber or plywood. A marketable mix of 1000 board feet of lumber products requires 1000 board feet of spruce and 4000 board feet of Douglas fir. Producing 1000 square feet of plywood requires 2000 board feet of spruce and 4000 board feet of fir. The company's timberland yields 32,000 board feet of spruce and 72,000 board feet of fir each season.

Sales commitments require that at least 5000 board feet of lumber and 12,000 board feet of plywood be produced during the season. The profit contributions are \$45 per 1000 board feet of lumber and \$60 per 1000 square feet of plywood.

- a) Express the problem as a linear programming model.
- b) Graph the problem and indicate the optimal solution on the graph.

Exercise 56 (Wagner) An electronics firm manufactures radio models A, B and C, which have profit contributions of 16, 30 and 50 respectively. Minimum production requirements are 20, 120 and 60 for the three models, respectively.

A dozen units of Model A requires 3 hours for manufacturing of components, 4 for assembling, and 1 for packaging. The corresponding figures for a dozen units of Model B are 3.5, 5, and 1.5, and for Model C are 5, 8, and 3. During the forthcoming week the company has available 120 plant-hours for manufacturing components, 160 for assembly, and 48 for packaging. Formulate this production planning problem as a linear programming problem.

Exercise 57 (Wagner) Process 1 in an oil refinery takes a unit feed of 1 bbl (barrel) of crude oil A and 3 bbl of crude B to make 50 gallons (gal) of gasoline X and 20 gal of gasoline Y. From a unit feed of 4 bbl crude A and 2 bbl crude B, process 2 makes 30 gal of X and 80 gal of Y. Let x_i be the number of units of feed type i ; e.g., $x_1 = 1$ indicates that process 1 uses 1 bbl crude A and 3 bbl crude B.

There are 120 bbl of crude A available and 180 bbl of crude B. Sales commitments call for at least 2800 gal of gasoline X and 2200 gal of Y. The unit profits of process 1 and 2 are p_1 and p_2 , respectively. Formulate an LP model.

Exercise 58 (Wagner) An air cargo firm has 8 aircraft of type 1, 15 of type 2 and 11 of type 3 available for today's flights. A type 1 craft can carry 45 tons, type 2, 7 tons and type 3, 5 tons. 20 tons of cargo are to be flown to city A and 28 tons to city B. Each plane makes at most one flight a day.

The costs of flying a plane from the terminal to each city are as follows.

	Type 1	Type 2	Type 3
City A	23	15	1.4
City B	58	20	3.8

Let x_i be the number of type i planes sent to A and y_i the number to B. Formulate an LP for this routing problem.

Exercise 59 (Wagner) A manufacturing firm produces widgets and distributes them to five wholesalers at a fixed delivered price of \$2.50 per unit. Sales forecasts indicate that monthly deliveries will be 2700, 2700, 9000, 4500 and 3600 widgets to wholesalers 1-5 respectively. The monthly production capacities are 4500, 9000 and 11,250 at plants 1, 2 and 3, respectively. The direct costs of producing each widget are \$2 at plant 1, \$1 at plant 2 and \$1.80 at plant 3.

The transport cost of shipping a widget from a plant to a wholesaler is given below.

Wholesaler	1	2	3	4	5
Plant 1	.05	.07	.11	.15	.16
Plant 2	.08	.06	.10	.12	.15
Plant 3	.10	.09	.09	.10	.16

Formulate an LP model for this production and distribution problem.

Exercise 60 (Wagner) Ft. Loudoun and Watts Bar are two large hydroelectric dams, the former upstream of the latter. The level of Watts Bar Lake must be kept within limits for recreational purposes, and the problem is to plan releases from Ft. Loudoun to do so. In reality this problem is solved simultaneously for numerous dams covering an entire watershed, but we focus on a single reservoir. There are also sophisticated models for predicting runoff into the reservoirs, but we will suppose that runoff is negligible. Thus any water entering Watts Bar Lake must be released through Ft. Loudoun Dam.

The planning period is 20 months. During month t , let x_t be the average water level of Watts Bar Lake before augmentation by water from Ft. Loudoun; $x_1 = 25$. Let y_t be the number of feet added to the average level in month t from Ft. Loudoun. L_t and U_t are the lower and upper bounds on the lake level in month t (more restrictive in summer). To model seepage, evaporation and hydroelectric release through Watts Bar Dam we suppose that Watts Bar Lake begins month $t+1$ at a level equal to .75 times the average level of the previous month (including the augmentation from Ft. Loudoun). The cost of water from Ft. Loudoun Lake is c_t for every foot added to the level of Watts Bar. Formulate the appropriate LP model. (In reality the model is a huge nonlinear program.)

Exercise 61 Red Dwarf Toasters needs to produce 1000 of their new "Talking Toaster". There are three ways this toaster can be produced: manually, semi-automatically, and robotically. Manual assembly requires 1 minute of skilled labor, 40 minutes of unskilled labor, and 3 minutes of assembly room time. The corresponding values for semiautomatic assembly are 4, 30, and 2; while those for robotic assembly are 8, 20, and 4. There are 4500 minutes of skilled labor, 36,000 minutes of unskilled labor, and 2700 minutes of assembly room time available for this product. The

total cost for producing manually is \$7/toaster; semiautomatically is \$8/toaster; and robotically is \$8.50/toaster.

(a) Formulate the problem of producing 1000 toasters at minimum cost meeting the resource requirements. Clearly define your variables, objective and constraints.

(b) Our union contract states that the amount of skilled labor time used is at least 10% of the total labor (unskilled plus skilled) time used. Update your formulation in (a) to handle this requirement.

(c) Any unused assembly floor time can be rented out at a profit of \$0.50/minute. Update your formulation to include this possibility.

Answers to Exercise 61:

(a) Let x_1 be the number of toasters produced manually, x_2 be the number produced semiautomatically, and x_3 be the number produced robotically.

The objective is to Minimize $7x_1 + 8x_2 + 8.5x_3$.

The constraints are:

$$x_1 + x_2 + x_3 = 1000 \text{ (produce enough toasters)}$$

$$x_1 + 4x_2 + 8x_3 \leq 4500 \text{ (skilled labor used less than or equal to amount available).}$$

$$40x_1 + 30x_2 + 20x_3 \leq 36000 \text{ (unskilled labor constraint)}$$

$$3x_1 + 2x_2 + 4x_3 \leq 2700 \text{ (assembly time constraint)}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \text{ (nonnegativity of production)}$$

(b) Add a constraint $x_1 + 4x_2 + 8x_3 \geq .1(41x_1 + 34x_2 + 28x_3)$

(c) Add a variable s_a to represent the assembly time slack. Add $+0.5s_a$ to the objective. Change the assembly time constraint to

$$3x_1 + 2x_2 + 4x_3 + s_a = 2700 \text{ (assembly time constraint)}$$

$$s_a \geq 0$$

