Solutions for Exercises 34-43

Solution to Exercise 34

(a)
\[
\begin{align*}
\text{MIN } & 10x + 2y^2 + z^2 + 8z \\
\text{subject to } & x + y + z = 100
\end{align*}
\]

(b) The Lagrangian associated with the formulation in part (a) is:
\[
L = 10x + 2y^2 + z^2 + 8z + \lambda(100 - x - y - z)
\]

(c) If we solve the Lagrangian:
\[
\begin{align*}
\frac{\partial L}{\partial x} &= 10 - \lambda = 0 \\
\frac{\partial L}{\partial y} &= 4y - \lambda = 0 \\
\frac{\partial L}{\partial z} &= 2z + 8 - \lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= 100 - x - y - z = 0
\end{align*}
\]

We get \( \lambda = 10 \) from the first equation. From the second equation, we get \( y = 2.5 \), and from the third equation we get \( z = 1 \). From the last equation we find \( x = 96.5 \).

(d) The marginal cost of producing one extra gold record is \( \lambda = 10 \).

(e) The formulation will be:
\[
\begin{align*}
\text{MIN } & 10x + 2y^2 + z^2 + 8z \\
\text{subject to } & x + y + z = 100 \\
& 4y + 2z = 60
\end{align*}
\]
The Lagrangian is:

\[ L = 10x + 2y^2 + z^2 + 8z + \lambda_1(100 - x - y - z) + \lambda_2(60 - 4y - 2z) \]

The set of equalities that must be solved for to find the optimal solution is:

\[
\begin{align*}
\frac{\partial L}{\partial x} &= 10 - \lambda_1 = 0 \\
\frac{\partial L}{\partial y} &= 4y - \lambda_1 - 4\lambda_2 = 0 \\
\frac{\partial L}{\partial z} &= 2z + 8 - \lambda_1 - 2\lambda_2 = 0 \\
\frac{\partial L}{\partial \lambda_1} &= 100 - x - y - z = 0 \\
\frac{\partial L}{\partial \lambda_2} &= 60 - 4y - 2z = 0
\end{align*}
\]

Solution to Exercise 35

(a)

MAX \[ 2x + y \]

subject to

\[ 4x^2 + y^2 = 8 \]

The Lagrangian associated with the formulation is:

\[ L = 2x + y + \lambda(8 - 4x^2 - y^2) \]

If we solve this Lagrangian:

\[
\begin{align*}
\frac{\partial L}{\partial x} &= 2 - 8x\lambda = 0 \Rightarrow x = \frac{1}{4\lambda} \\
\frac{\partial L}{\partial y} &= 1 - 2y\lambda = 0 \Rightarrow y = \frac{1}{2\lambda} \\
\frac{\partial L}{\partial \lambda} &= 8 - 4x^2 - y^2 = 0 \Rightarrow 8 - 4\left(\frac{1}{4\lambda}\right)^2 - \left(\frac{1}{2\lambda}\right)^2 = 0
\end{align*}
\]

We get \( x = 1, \ y = 2, \ \lambda = 0.25 \)

(b) If \( Q \) increases by \( \Delta \), the optimal objective function value increases by \( \lambda \Delta \) \Rightarrow the change in optimal objective function will be \( 0.25 \times (8.4 - 8) = 0.1 \).
Solution to Exercise 36

(a)

\[
\text{MAX } \ln x + 2\ln y + 3\ln z \\
\text{subject to } x + y + z = 60
\]

The Lagrangian associated with the formulation is:

\[
L = \ln x + 2\ln y + 3\ln z + \lambda(60 - x - y - z)
\]

If we solve this Lagrangian:

\[
\frac{\partial L}{\partial x} = \frac{1}{x} - \lambda = 0 \rightarrow x = \frac{1}{\lambda} \\
\frac{\partial L}{\partial y} = \frac{2}{y} - \lambda = 0 \rightarrow y = \frac{2}{\lambda} \\
\frac{\partial L}{\partial z} = \frac{3}{z} - \lambda = 0 \rightarrow z = \frac{3}{\lambda} \\
\frac{\partial L}{\partial \lambda} = 60 - x - y - z = 0 \rightarrow 60 - \frac{1}{\lambda} - \frac{2}{\lambda} - \frac{3}{\lambda} = 0
\]

We get \( x = 10, y = 20, z = 30, \lambda = 0.1 \)

(b) If \( Q \) increases by \( \Delta \), the optimal objective function value increases by \( \lambda \Delta \) \rightarrow the change in optimal objective function will be \( 0.1 \times (65 - 60) = 0.5 \).

Solution to Exercise 37

We assume in this exercise that \( x_1, x_2, x_3 > 0 \); hence the problem can be written as

\[
\text{MAX } 2\ln x_1 + 3\ln x_2 + 3\ln x_3 \\
\text{subject to } x_1 + 2x_2 + 2x_3 = 10
\]

The Lagrangian associated with the formulation is:

\[
L = 2\ln x_1 + 3\ln x_2 + 3\ln x_3 + \lambda(10 - x_1 - 2x_2 - 2x_3)
\]
If we solve this Lagrangian:

\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= \frac{2}{x_1} - \lambda = 0 \rightarrow x_1 = \frac{2}{\lambda} \\
\frac{\partial L}{\partial x_2} &= \frac{3}{x_2} - 2\lambda = 0 \rightarrow x_2 = \frac{3}{2\lambda} \\
\frac{\partial L}{\partial x_3} &= \frac{3}{x_3} - 2\lambda = 0 \rightarrow x_3 = \frac{3}{2\lambda} \\
\frac{\partial L}{\partial \lambda} &= 10 - x_1 - 2x_2 - 2x_3 = 0 \rightarrow 10 - \frac{2}{\lambda} - 2 \times \frac{3}{2\lambda} - 2 \times \frac{3}{2\lambda} = 0
\end{align*}
\]

We get \( x_1 = \frac{20}{8} \), \( x_2 = \frac{30}{16} \), \( x_3 = \frac{30}{16} \), \( \lambda = 0.8 \)

**Solution to Exercise 38**

Let \((x_1, x_2)\) and \((y_1, y_2)\) be the two points closest to \((1, 0)\). The distances will be \(d_1\) and \(d_2\) for these two points, respectively:

\[
\begin{align*}
d_1^2 &= (x_1 - 1)^2 + x_2^2 \\
d_2^2 &= (y_1 - 1)^2 + y_2^2
\end{align*}
\]

The minimization formulation for this problem is:

\[
\begin{align*}
\text{MIN } (x_1 - 1)^2 + x_2^2 \\
\text{subject to } x_1^2 + 4x_2^2 &= 4
\end{align*}
\]

The Lagrangian associated with the formulation is:

\[
L = (x_1 - 1)^2 + x_2^2 + \lambda(4 - x_1^2 - 4x_2^2)
\]

If we solve this Lagrangian:

\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 2(x_1 - 1) - 2\lambda x_1 = 0 \rightarrow x_1 = \frac{1}{1 - \lambda} \\
\frac{\partial L}{\partial x_2} &= 2x_2 - 8\lambda x_2 = 0 \rightarrow x_2(1 - 4\lambda) = 0 \\
\frac{\partial L}{\partial \lambda} &= 4 - x_1^2 - 4x_2^2 = 0
\end{align*}
\]
From second equation, \( x_2 = 0 \) or \( \lambda = 1/4 \). If \( x_2 = 0 \), \( x_1 = 2 \) or \( x_1 = -2 \). \( d_1^2 = 1 \) when \( x_1 = 2 \), and \( d_1^2 = 9 \) when \( x_1 = -2 \). If \( \lambda = 1/4 \), \( x_1 = 3/4 \) and \( x_2 = \sqrt{5/9} \) or \( x_2 = -\sqrt{5/9} \). In this case, \( d_1^2 = 6/9 \) for both \( x_2 \) values. So, the closest points are: \( (4/3, \sqrt{5/9}) \) and \( (4/3, -\sqrt{5/9}) \).

**Solution to Exercise 39**

(a)

\[
\text{MIN } x_1^2 + x_2^2 + x_3^2
\]
subject to
\[
x_1 + x_2 + x_3 = b
\]
The Lagrangian associated with the formulation is:
\[
L = x_1^2 + x_2^2 + x_3^2 + \lambda(b - x_1 - x_2 - x_3)
\]

If we solve this Lagrangian:

\[
\begin{align*}
\frac{\partial L}{\partial x_1} & = 2x_1 - \lambda = 0 \rightarrow x_1 = \frac{\lambda}{2} \\
\frac{\partial L}{\partial x_2} & = 2x_2 - \lambda = 0 \rightarrow x_2 = \frac{\lambda}{2} \\
\frac{\partial L}{\partial x_3} & = 2x_3 - \lambda = 0 \rightarrow x_3 = \frac{\lambda}{2} \\
\frac{\partial L}{\partial \lambda} & = b - x_1 - x_2 - x_3 = 0 \rightarrow b - \frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 0
\end{align*}
\]

We get \( x_1 = x_2 = x_3 = \frac{b}{3}, \lambda = \frac{2b}{3} \).

(b)

\[
\text{MAX } \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}
\]
subject to
\[
x_1 + x_2 + x_3 = b
\]
The Lagrangian associated with the formulation is:
\[
L = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \lambda(b - x_1 - x_2 - x_3)
\]

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If we solve this Lagrangian:

\[
\frac{\partial L}{\partial x_1} = \frac{1}{2\sqrt{x_1}} - \lambda = 0 \rightarrow x_1 = \frac{1}{4\lambda^2} \\
\frac{\partial L}{\partial x_2} = \frac{1}{2\sqrt{x_2}} - \lambda = 0 \rightarrow x_2 = \frac{1}{4\lambda^2} \\
\frac{\partial L}{\partial x_3} = \frac{1}{2\sqrt{x_3}} - \lambda = 0 \rightarrow x_3 = \frac{1}{4\lambda^2} \\
\frac{\partial L}{\partial \lambda} = b - x_1 - x_2 - x_3 = 0 \rightarrow b - \frac{1}{4\lambda^2} - \frac{1}{4\lambda^2} - \frac{1}{4\lambda^2} = 0
\]

We get \(x_1 = x_2 = x_3 = \frac{b}{3}, \lambda = \sqrt{\frac{3}{4b}}\).

(c) We assume that \(c_1, c_2, c_3 > 0\) \(\text{MIN } c_1x_1^2 + c_2x_2^2 + c_3x_3^2\)

subject to

\[x_1 + x_2 + x_3 = b\]

The Lagrangian associated with the formulation is:

\[L = c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + \lambda(b - x_1 - x_2 - x_3)\]

If we solve this Lagrangian:

\[
\frac{\partial L}{\partial x_1} = 2c_1x_1 - \lambda = 0 \rightarrow x_1 = \frac{\lambda}{2c_1} \\
\frac{\partial L}{\partial x_2} = c_2x_2 - \lambda = 0 \rightarrow x_2 = \frac{\lambda}{2c_2} \\
\frac{\partial L}{\partial x_3} = 2c_3x_3 - \lambda = 0 \rightarrow x_3 = \frac{\lambda}{2c_3} \\
\frac{\partial L}{\partial \lambda} = b - x_1 - x_2 - x_3 = 0 \rightarrow b - \frac{\lambda}{2c_1} - \frac{\lambda}{2c_2} - \frac{\lambda}{2c_3} = 0
\]

\[x_1 = \frac{b}{2c_1} \left( \frac{1}{2c_1} + \frac{1}{2c_2} + \frac{1}{2c_3} \right), \quad x_2 = \frac{b}{2c_2} \left( \frac{1}{2c_1} + \frac{1}{2c_2} + \frac{1}{2c_3} \right), \quad x_3 = \frac{b}{2c_3} \left( \frac{1}{2c_1} + \frac{1}{2c_2} + \frac{1}{2c_3} \right)\]

(d) \(\text{MIN } x_1^2 + x_2^2 + x_3^2\)
subject to
\[
\begin{align*}
x_1 + x_2 &= b_1 \\
x_2 + x_3 &= b_2
\end{align*}
\]

The Lagrangian associated with the formulation is:
\[
L = x_1^2 + x_2^2 + x_3^2 + \lambda_1 (b - x_1 - x_2) + \lambda_2 (b - x_2 - x_3)
\]

If we solve this Lagrangian:
\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 2x_1 - \lambda_1 = 0 \implies x_1 = \frac{\lambda_1}{2} \\
\frac{\partial L}{\partial x_2} &= 2x_2 - \lambda_1 - \lambda_2 = 0 \implies x_2 = \frac{\lambda_1 + \lambda_2}{2} \\
\frac{\partial L}{\partial x_3} &= 2x_3 - \lambda_2 = 0 \implies x_3 = \frac{\lambda_2}{2} \\
\frac{\partial L}{\partial \lambda_1} &= b_1 - x_1 - x_2 = 0 \\
\frac{\partial L}{\partial \lambda_2} &= b_2 - x_2 - x_3 = 0
\end{align*}
\]

We get
\[
\begin{align*}
x_1 &= \frac{2b_1 - b_2}{3}, \\
x_2 &= \frac{b_1 + b_2}{3}, \\
x_3 &= \frac{2b_1 - b_2}{3}, \\
\lambda_1 &= \frac{4b_1 - 2b_2}{3}, \\
\lambda_2 &= \frac{4b_2 - 2b_1}{3}
\end{align*}
\]

**Solution to Exercise 40**

MAX \(by - x^4\)

subject to
\[
x^2 + cy = a
\]

The Lagrangian associated with the formulation is:
\[
L = by - x^4 + \lambda(a - x^2 - cy)
\]

If we solve this Lagrangian:
\[
\begin{align*}
\frac{\partial L}{\partial x} &= -4x^3 - 2x\lambda = 0 \implies x = \sqrt[4]{\frac{-\lambda}{2}} \\
\frac{\partial L}{\partial y} &= b - c\lambda = 0 \implies \lambda = \frac{b}{c} \\
\frac{\partial L}{\partial \lambda} &= a - x^2 - cy = 0
\end{align*}
\]
When $a$ is increased to $a + \frac{5}{100}a$, the optimal value would increase by $\lambda \times 0.05a$ which is $0.05a^2$.

**Solution to Exercise 41**

(a)

$$\text{MAX } r_1 x_1 + r_2 x_2$$

subject to

$$\sigma^2 x_1^2 + \rho x_1 x_2 + \sigma^2 x_2^2 = s^2$$

The Lagrangian associated with the formulation is:

$$L = r_1 x_1 + r_2 x_2 + \lambda(s^2 - \sigma^2 x_1^2 - \rho x_1 x_2 - \sigma^2 x_2^2)$$

If we solve this Lagrangian:

$$\begin{align*}
\frac{\partial L}{\partial x_1} &= r_1 - 2\lambda \sigma^2 x_1 - \lambda \rho x_2 = 0 \\
\frac{\partial L}{\partial x_2} &= r_2 - 2\lambda \sigma^2 x_2 - \lambda \rho x_1 = 0 \\
\frac{\partial L}{\partial \lambda} &= s^2 - \sigma^2 x_1^2 - \rho x_1 x_2 - \sigma^2 x_2^2 = 0
\end{align*}$$

By using the first two equations:

$$\begin{align*}
x_1 &= \frac{s}{\sqrt{\sigma^2 \left( \frac{r_1 \rho - 2 \sigma_2}{r_2 - 2 \sigma_1} \right)^2 + \rho \left( \frac{r_1 \rho - 2 \sigma_2}{r_2 - 2 \sigma_1} \right) + \sigma^2}} \\
x_2 &= \frac{s}{\sqrt{\sigma^2 \left( \frac{r_2 \rho - 2 \sigma_1}{r_1 - 2 \sigma_2} \right)^2 + \rho \left( \frac{r_2 \rho - 2 \sigma_1}{r_1 - 2 \sigma_2} \right) + \sigma^2}}
\end{align*}$$

(b) If $r_1 = r_2 = r$:

$$\begin{align*}
x_1 &= \frac{s}{\sqrt{2\sigma^2 + \rho}} \\
x_2 &= \frac{s}{\sqrt{2\sigma^2 + \rho}}
\end{align*}$$
Solution to Exercise 42

\[ \text{MIN } \pi^2 r^4 + \pi^2 r^2 h^2 \]

subject to \( \frac{1}{3} \pi r^2 h = V_0 \)

The Lagrangian associated with the formulation is:
\[
L = \pi^2 r^4 + \pi^2 r^2 h^2 + \lambda (V_0 - 1/3 \pi r^2 h)
\]

If we solve this Lagrangian:

\[
\frac{\partial L}{\partial r} = 4 \pi^2 r^3 + 2 \pi^2 r h^2 - \frac{2}{3} \pi r h = 0
\]
\[
\frac{\partial L}{\partial h} = 2 \pi^2 r^2 h - \frac{1}{3} \pi r^2 \lambda = 0 \rightarrow r^2 (2 \pi h - \frac{\lambda}{3}) = 0 \rightarrow h = \frac{\lambda}{6 \pi} \text{ (use Hint)}
\]
\[
\frac{\partial L}{\partial \lambda} = V_0 - 1/3 \pi r^2 h = 0 \rightarrow r^2 = \frac{3V_0}{\pi h}
\]

By using the first equation:
\[
h = \left( \frac{12V_0}{4 \pi^3 - 2 \pi^2} \right)^{1/3}
\]
\[
r = \frac{3V_0}{\pi \left( \frac{12V_0}{4 \pi^3 - 2 \pi^2} \right)^{1/3}}
\]

Solution to Exercise 43

(a)

\[ \text{MAX } 400p_A + 200p_B - 2p_A^2 - p_B^2 + 2p_A p_B \]

subject to
\[
-p_A \quad -p_B \quad \leq \quad -400
\]
\[
-p_B \quad \leq \quad -600
\]

The Lagrangian associated with the formulation is:
\[
L = 400p_A + 200p_B - 2p_A^2 - p_B^2 + 2p_A p_B + \mu_1 (-400 + p_A + p_B) + \mu_2 (-600 + p_B)
\]
which gives the optimality conditions

\[
\begin{align*}
400 - 4p_A + 2p_B + \mu_1 &= 0 \\
200 - 2p_B + 2p_A + \mu_1 + \mu_2 &= 0 \\
\mu_1(-400 + p_A + p_B) &= 0 \\
\mu_2(-600 + p_B) &= 0 \\
-p_A - p_B &\leq -400 \\
-p_B &\leq -600
\end{align*}
\]

Since there are two complimentary conditions, there are 4 cases to check:

- \( \mu_1 = 0, \mu_2 = 0 \) : gives \( p_A = 300, p_B = 400 \) which is not feasible
- \( \mu_1 = 0, -600 + p_B = 0 \) : gives \( p_A = 400, p_B = 600 \) which is OK
- \( 400 + p_A + p_B = 0, -600 + p_B = 0 \) : gives \( p_A = -200, p_B = 600 \) which is not feasible
- \( 400 + p_A + p_B = 0, \mu_2 = 0 \) : gives \( p_B = -300 \) which is not feasible

Therefore, the optimal solution is at \( p_A = 400, p_B = 600 \).

(b) As \( \mu_1 = 0 \), the company will not be willing to pay any extra money on another hour of labor. As \( \mu_2 = 200 \), they will be willing to pay at maximum 200 for another unit of raw material.